

# DUAL $n_1$ -APPELL-LIKE SYSTEMS IN INFINITE-DIMENSIONAL ANALYSIS

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The biorthogonal approach to non-Gaussian infinite-dimensional analysis [1] is generalizing and developing in, at least, three directions. The first one deals with the biorthogonal decomposition of  $L_2(N', \mu)$ , where  $\mu$  is a probability measure on co-nuclear space  $N'$ , by means of dual Appell systems, which consist of Appell polynomials  $\langle P_n(x), \varphi^{(n)} \rangle$ , for which

$$\int_{N'} \langle P_n(x), \varphi^{(n)} \rangle \mu(dx) = 0, \quad n \in \mathbb{N}, \quad (1)$$

and dual set of distributions (see [2–5]).

The second direction is biorthogonal analysis, connected with hypergroups ([6]) and generalized translation operators ([7]), where the Appell polynomials are substituted by Appell or Delsarte characters, that, generally speaking, are not polynomials.

The third direction deals with the generalized Appell ([8,9]) and Appell-like ([10,11]) systems. These systems do not satisfy (1) and allow to study pseudodifferential equations on Kondratiev spaces, see [8–11].

In this paper we will generalize the construction and results of [10,11] on the wider class of generating functions. The necessity of such generalization arose because requirements of applications. For example, this generalization gives the possibility to construct biorthogonal analysis with respect to measures, which are connected with symmetric Brownian motions [12].

1. Let  $\mathcal{H}$  be a separable real Hilbert space,  $N$ —separable nuclear Fréchet space, which is embedded in  $\mathcal{H}$  topologically (i.e. densely and continuously). We denote by  $N'$  the space, which is dual to  $N$  with respect to  $\mathcal{H}$ . Further we will use representations  $N = \text{pr} \lim_{p \in \mathbb{N}} \mathcal{H}_p$ ,  $N' = \text{ind} \lim_{p \in \mathbb{N}} \mathcal{H}_{-p}$  where  $\mathcal{H}_p$  are Hilbert spaces such that  $\forall p \exists p' > p$ :  $\mathcal{H}_{p'} \subseteq \mathcal{H}_p$ , where embedding is of Hilbert–Schmidt class,  $\mathcal{H}_{-p}$  is dual space of the chain  $\mathcal{H}_p \subseteq \mathcal{H} \subseteq \mathcal{H}_{-p}$  (see, for example, [4]). Let us denote by  $|\cdot|_p$  the norm in  $\mathcal{H}_p$  ( $p \in \mathbb{Z}$ ),  $\langle x, \theta \rangle$ —dual pairing between  $x \in N'$  and  $\theta \in N$ , which is given by extension of the inner product in  $\mathcal{H}$ . We will preserve these notations for tensor powers of spaces and complexifications. We will denote by subscript  $\mathbb{C}$  the complexification of spaces, for example  $N_{\mathbb{C}}$  is the complexification of  $N$ . Let  $\text{Hol}_0(N_{\mathbb{C}})$  be the algebra of germs of functions  $\gamma : N_{\mathbb{C}} \rightarrow \mathbb{C}$ , holomorphic in  $0 \in N_{\mathbb{C}}$ . We denote by  $\hat{\otimes}$  the symmetric tensor product.

2. Fix  $n_1 \in \mathbb{N}$ . In what follows we denote  $\tilde{n} := nn_1$ ,  $n \in \mathbb{N}$ . Let  $\chi : \mathbb{C} \rightarrow \mathbb{C}$  be entire function with  $\chi(0) = 0$ , and its Taylor decomposition is  $\chi(s) = \sum_{n=0}^{\infty} \frac{\chi_{\tilde{n}}}{\tilde{n}!} s^{\tilde{n}}$ ,

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$s \in \mathbb{C}$ ,  $\chi_{\tilde{n}} \neq 0$ ,  $\forall n \in \mathbb{Z}_+$ . Let  $\gamma \in \text{Hol}_0(N_{\mathbb{C}})$ ,  $\gamma(0) \neq 0$  and in some neighborhood of  $0 \in N_{\mathbb{C}}$   $\gamma(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \gamma_{\tilde{n}}, \theta^{\otimes \tilde{n}} \rangle$ ,  $\theta \in N_{\mathbb{C}}$ ,  $\gamma_{\tilde{n}} \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}}$  (it is not hard to prove, that if  $\gamma^{(1)}$  and  $\gamma^{(2)}$  satisfy these conditions, then  $\gamma^{(1)}(\theta)/\gamma^{(2)}(\theta)$  also satisfy them). We put  $\chi^{\gamma}(\theta; z) := \gamma(\theta)\chi(\langle z, \theta \rangle)$ ,  $z \in N_{\mathbb{C}}'$ ,  $\theta \in N_{\mathbb{C}}$ . One can decompose  $\chi^{\gamma}(\cdot; z)$  in Taylor series and obtain by kernel theorem the representation

$$\chi^{\gamma}(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{\tilde{n}!} \langle P_n^{X, \gamma}(z), \theta^{\otimes \tilde{n}} \rangle, \quad P_n^{X, \gamma}(z) \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}}.$$

**Definition 1.** We will call  $\mathbf{P}^{X, \gamma} := \{ \langle P_n^{X, \gamma}(\cdot), \varphi^{(\tilde{n})} \rangle : \varphi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}}, n \in \mathbb{Z}_+ \}$  the (generalized)  $n_1$ -Appell-like polynomial system.

Particularly, 1-Appell-like polynomials are Appell-like polynomials, see [10,11] and references therein.

The properties of  $n_1$ -Appell-like polynomials, as it easy can be proved, are analogous to the same of Appell-like polynomials. So, for example,

$$P_n^{X, \gamma^{(1)}}(z) = \sum_{m=0}^n C_n^{\tilde{m}} P_m^{X, \gamma^{(2)}}(z) \widehat{\otimes} \widetilde{\gamma_{n-m}}, \quad (2)$$

where  $\gamma^{(1)}, \gamma^{(2)}$  satisfy the conditions above to  $\gamma$ ,  $\widetilde{\gamma_{n-m}} \in N_{\mathbb{C}}^{\prime \otimes (n-m)}$  from decomposition  $\gamma^{(1)}(\theta)/\gamma^{(2)}(\theta) = \sum_{n=0}^{\infty} \frac{1}{\tilde{n}!} \langle \widehat{\gamma_{\tilde{n}}}, \theta^{\otimes \tilde{n}} \rangle$ .

Let  $\mathcal{P}_{X, \gamma} := \{ \varphi(x) = \sum_{n=0}^{M_{\varphi}} \langle P_n^{X, \gamma}(x), \varphi^{(\tilde{n})} \rangle : x \in N', \varphi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}} \}$ . It follows from (2) that  $\mathcal{P}_{X, \gamma}$  does not depend on  $\gamma$ . The dependence on  $\chi$  connected only with  $n_1$ , therefore for  $\mathcal{P}_{X, \gamma}$  we can accept the notation  $\mathcal{P}_{n_1}$ . We choose the topology in  $\mathcal{P}_{n_1}$  such that it be topologically isomorphic to topological direct sum of tensor powers of  $N_{\mathbb{C}} \bigoplus_{n=0}^{\infty} N_{\mathbb{C}}^{\prime \otimes \tilde{n}}$  via the coefficients of  $n_1$ -Appell-like polynomials. Note, that if  $n_1 \neq 1$ , then  $\mathcal{P}_{n_1}$  does not coincide with the set  $\mathcal{P} = \mathcal{P}_1$  of all continuous polynomials on  $N'$ , because  $\mathcal{P}_{n_1}$  contains only polynomials with  $n_1$ -divisible powers.

Let  $(\mathcal{H}_p)_{q, X, \gamma}^1$  be the completion of  $\mathcal{P}_{n_1}$  with respect to the norm  $\| \cdot \|_{p, q, X, \gamma}$ , which is defined for  $\varphi(x) = \sum_{n=0}^{\infty} \langle P_n^{X, \gamma}(x), \varphi^{(\tilde{n})} \rangle$ ,  $x \in N'$ ,  $\varphi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}}$  by equality  $\| \varphi \|_{p, q, X, \gamma}^2 := \sum_{n=0}^{\infty} (\tilde{n}!)^2 2^{q\tilde{n}} |\varphi^{(\tilde{n})}|_p^2$ ;  $(N)_{X, \gamma}^1 := \text{pr} \lim_{p, q \in \mathbb{N}} (\mathcal{H}_p)_{q, X, \gamma}^1$ .

**Theorem 1.** For  $\chi, \gamma$  under conditions above the spaces  $(N)_{X, \gamma}^1$  are invariant with respect to  $\gamma$ , i.e.  $(N)_{X, \gamma}^1 = (N)_{\chi}^1$ . If there exists positive nondecreasing function  $\Psi$  of real argument such that  $|\chi(z)| \leq \Psi(|z|)$ ,  $\forall z \in \mathbb{C}$ , then each element from  $(N)_{\chi}^1$  is restriction on  $N'$  of some entire on  $N'_{\mathbb{C}}$  function. Moreover, if there exist  $C_0 > 0$  and  $C_1 > 0$  such that  $\Psi(r) \leq C_0 \exp\{C_1 r\}$ , then  $(N)_{\chi}^1 \subseteq (N)^1 := (N)_{\exp}^1$ .

The proof of this theorem is based on formula (2), for details see [11].  $\square$

**Example.** Let us consider the function  $\Lambda_s(u) := \sum_{m=0}^{\infty} \frac{u^{2m}}{(2m)!} \cdot \left(-\frac{1}{4}\right)^m \frac{(2m)!s!}{m!(s+m)!}$ ,  $u \in \mathbb{C}$ ,  $s \in [-\frac{1}{2}, \infty)$ , where  $s! := \Gamma(s+1)$  ( $\Lambda_s$  is related to random walks with spherical symmetry, see [12]). It is easy to prove, that  $\Lambda_s(u)$  satisfies all conditions of Theorem 1:  $n_1 = 2$ ,  $\Psi(r) = e^r$ .

We define on  $\mathcal{P}_{n_1}$  pseudodifferential operator  $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle$ ,  $\Phi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \widehat{\otimes} \tilde{n}}$ , setting on monomials

$$\begin{aligned} & \langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle \langle x^{\otimes \tilde{m}}, \varphi^{(\tilde{m})} \rangle \\ & := 1_{\{m \geq n\}} \frac{\tilde{m}! \chi_{\tilde{m}-n}}{(m-n)! \chi_{\tilde{m}}} \langle x^{\otimes (\tilde{m}-n)} \widehat{\otimes} \Phi^{(\tilde{n})}, \varphi^{(\tilde{m})} \rangle, \varphi^{(\tilde{m})} \in N_{\mathbb{C}}^{\widehat{\otimes} \tilde{m}}, x \in N', \end{aligned}$$

and extending by linearity (here  $1_{\{m \geq n\}}$  is the indicator of  $\{m \geq n\}$ ).

**Theorem 2.** *The  $n_1$ -Appell-like polynomials are generalized powers with respect to  $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle$ , i.e.*

$$\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle \langle P_{\tilde{m}}^{\chi, \gamma}(x), \varphi^{(\tilde{m})} \rangle = 1_{\{m \geq n\}} \frac{\tilde{m}!}{(m-n)!} \langle P_{\tilde{m}-n}^{\chi, \gamma}(x) \widehat{\otimes} \Phi^{(\tilde{n})}, \varphi^{(\tilde{m})} \rangle.$$

The proof uses (2) and definition of  $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle$ .  $\square$

**Corollary.** *For all  $\Phi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \widehat{\otimes} \tilde{n}}$   $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle \in \mathcal{L}(\mathcal{P}_{n_1}, \mathcal{P}_{n_1})$ .*

3. Let  $H$  be a separable real Hilbert space such that  $\mathcal{P}_{n_1}$  is embedded in  $H$  and for  $\varphi \in \mathcal{P}_{n_1}$   $\|\varphi\|_H = 0 \Rightarrow \varphi(x) = 0 \forall x \in N'$ . Suppose also that there exist  $C > 0$ ,  $K > 0$ ,  $p \in \mathbb{N}$  such that  $\|P_{\tilde{n}}^{\chi, 1}(\cdot)_{-p}\|_H \leq \tilde{n}! C^{\tilde{n}} K$ , and embedding  $\mathcal{P}_{n_1} \hookrightarrow H$  is topological.

*Remark.* Note, that in the case  $n_1 = 1$  the main example of  $H$  is  $L_2(N', \mu)$ , where  $\mu$  is analytic nondegenerate probability measure (see [4,5,8–11]). But in the case  $n_1 \neq 1$  such spaces can turn out too wide for  $H$ . For example, if  $\mu$ —smooth measure with second analyticity condition (see [2]), then  $\mathcal{P}_{n_1}$  is not dense in  $L_2(N', \mu)$ , if  $n_1 \neq 1$ . But in the similar cases one can use the completion of  $\mathcal{P}_{n_1}$  with respect to the norm of  $L_2(N', \mu)$  as  $H$ .

Let  $\mathcal{P}'_{n_1, H}$  be the space, which is dual to  $\mathcal{P}_{n_1}$  with respect to  $H$ . Let  $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle_H^* \in \mathcal{L}(\mathcal{P}'_{n_1, H}, \mathcal{P}'_{n_1, H})$  be the operator, which is dual to  $\langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle$ , i.e. for all  $\Phi \in \mathcal{P}'_{n_1, H}$ ,  $\varphi \in \mathcal{P}_{n_1}$   $\langle \langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle_H^* \Phi, \varphi \rangle_H = \langle \langle \Phi, \langle \Phi^{(\tilde{n})}, D_{\chi}^{\otimes \tilde{n}} \rangle \varphi \rangle_H$ , where  $\langle \cdot, \cdot \rangle_H$  is the dual pairing between  $\mathcal{P}'_{n_1, H}$  and  $\mathcal{P}_{n_1}$ , which is given by extension of the inner product in  $H$ . One can prove that there exist  $p' = p'(\chi, \gamma)$ ,  $q' = q'(\chi, \gamma)$  such that for all  $p \geq p'$ ,  $q \geq q'$   $(\mathcal{H}_p)_{q, \chi, \gamma}^1 \hookrightarrow H$  topologically. Therefore,  $(N)_{\chi}^1 \hookrightarrow H$  topologically. Let  $(N)_{\chi, H}^{-1}$  be the space, which is dual to  $(N)_{\chi}^1$  with respect to  $H$ . We define on  $(N)_{\chi}^1$  the linear functional  $\delta_z$ ,  $z \in N'_{\mathbb{C}}$ , by formula  $\langle \delta_z, \varphi \rangle_H := \varphi(z)$ ,  $\varphi \in (N)_{\chi}^1$ . It is not hard to prove that  $\delta_z \in (N)_{\chi, H}^{-1}$ . Let  $\gamma$  be such as above. We put

$$Q_{H, \tilde{m}}^{\chi, \gamma}(\Phi^{(\tilde{m})}; \cdot) := \sum_{k=0}^{\infty} \frac{1}{k!} (\langle \Phi^{(\tilde{m})} \widehat{\otimes} \tilde{\gamma}_{\tilde{k}}, D_{\chi}^{\otimes (\tilde{m} + \tilde{k})} \rangle_H^* \delta_0)(\cdot) \in \mathcal{P}'_{n_1, H}, \quad (3)$$

where  $\Phi^{(\tilde{m})} \in N_{\mathbb{C}}^{\prime \widehat{\otimes} \tilde{m}}$ ,  $\tilde{\gamma}_{\tilde{k}} \in N_{\mathbb{C}}^{\prime \widehat{\otimes} \tilde{k}}$  from decomposition  $1/\gamma(\theta) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \tilde{\gamma}_{\tilde{k}}, \theta^{\otimes \tilde{k}} \rangle$ ,  $\theta \in N_{\mathbb{C}}$ .

**Theorem 3** (biorthogonality w.r.t.  $H$ ). *The generalized functions  $Q_{H, \tilde{m}}^{\chi, \gamma}(\Phi^{(\tilde{m})}; \cdot)$  are biorthogonal to  $\mathbf{P}^{\chi, \gamma}$ -system, i.e.*

$$\langle Q_{H, \tilde{m}}^{\chi, \gamma}(\Phi^{(\tilde{m})}; \cdot), \langle P_{\tilde{n}}^{\chi, \gamma}(\cdot), \varphi^{(\tilde{n})} \rangle \rangle_H = \delta_{mn} \tilde{n}! \langle \Phi^{(\tilde{n})}, \varphi^{(\tilde{n})} \rangle, \Phi^{(\tilde{m})} \in N_{\mathbb{C}}^{\prime \widehat{\otimes} \tilde{m}}, \varphi^{(\tilde{n})} \in N_{\mathbb{C}}^{\widehat{\otimes} \tilde{n}}.$$

The proof is based on (2), (3) and definition of  $\delta_0$ .  $\square$

**Theorem 4.** Each generalized function  $\Phi \in \mathcal{P}'_{n_1, H}$  can be represented in the form of

$$\Phi(\cdot) = \sum_{m=0}^{\infty} Q_{H, \tilde{m}}^{\chi, \gamma}(\Phi^{(\tilde{m})}; \cdot), \quad \Phi^{(\tilde{m})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{m}}, \quad (4)$$

where the sequence of kernels  $\{\Phi^{(\tilde{m})}\}_{m=0}^{\infty}$  is uniquely determined by  $\Phi$ . Conversely, each sequence  $\{\Phi^{(\tilde{m})}\}_{m=0}^{\infty}$ ,  $\Phi^{(\tilde{m})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{m}}$  defines the generalized function  $\Phi \in \mathcal{P}'_{n_1, H}$  by formula (4).

The proof is the same as the proof of corresponding statement in [11].  $\square$

**Definition 2.** The system of  $n_1$ -Appell-like polynomials  $\mathbf{P}^{\chi, \gamma}$  and the family of generalized functions  $Q_{H, \tilde{m}}^{\chi, \gamma}(\Phi^{(\tilde{m})}; \cdot)$ ,  $\Phi^{(\tilde{m})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{m}}$ , form dual  $n_1$ -Appell-like system.

Let us introduce an analog of  $S$ -transform (see [2–11]). For  $\Phi \in (N)_{\chi, H}^{-1}$  we put

$$(S_{\chi, \gamma, H}\Phi)(\theta) := \langle\langle \Phi(\cdot), \chi^{\gamma}(\theta; \cdot) \rangle\rangle_H, \quad \theta \in N_{\mathbb{C}}.$$

One can prove, that this definition is correct.

**Theorem 5** (*characterization theorem*).  $S_{\chi, \gamma, H}$ -transform is topological isomorphism from  $(N)_{\chi, H}^{-1}$  to subspace of  $\text{Hol}_0(N_{\mathbb{C}})$ , which consist of functions  $F$ , which can be written in the form of  $F(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(\tilde{n})}, \theta^{\otimes \tilde{n}} \rangle$ ,  $\Phi^{(\tilde{n})} \in N_{\mathbb{C}}^{\prime \otimes \tilde{n}}$ .

The proof is analogous to the proof of characterization theorem in [11].  $\square$

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